Set Theory Definitions

This handout lists some terms and formal definitions from set theory that might be useful for the first problem set.

Set Membership, Equality, and Subsets

An element of a set is an object contained within that set. For example, we have $1 \in \{1, 2, 3\}$ and $\emptyset \in \{\emptyset\}$, but $1 \notin \emptyset$ and $1 \notin \{2, 3\}$.

Two sets are equal iff they contain the same elements. For example, we have $\{1, 2\} = \{2, 1\}$ and $\{ \emptyset \} = \{ \emptyset \}$. However, $\{ \emptyset \} \neq \{ \{\emptyset\} \}$, because each set contains an element the other does not. A set and a non-set are never equal; in particular, this means $x \neq \{x\}$ for any x.

A set A is a subset of a set B (denoted $A \subseteq B$) iff every element of A is also an element of B:

$$\mathbb{N} \subseteq \mathbb{Z} \qquad \{1, 2, 3\} \subseteq \{1, 2, 3, 4\} \qquad \{1\} \subseteq \{1, \{1\}, \{\{1\}\}\}$$

A set A is a strict subset of a set B (denoted $A \subset B$) iff $A \subseteq B$ and $A \neq B$.

Set Operations

The set $\{ x \mid \textit{some property of } x \}$ is the set of all x's satisfying the given property. Formally, we have that $w \in \{ x \mid \textit{some property of } x \}$ iff that property holds for w.

The set $A \cup B$ is the set $\{ x \mid x \in A \text{ or } x \in B \}$. $x \in A \cup B$ iff $x \in A$ or $x \in B$.

The set $A \cap B$ is the set $\{ x \mid x \in A \text{ and } x \in B \}$. $x \in A \cap B$ iff $x \in A$ and $x \in B$.

The set $A - B$ is the set $\{ x \mid x \in A \text{ and } x \notin B \}$. This set is also sometimes denoted $A \setminus B$.

The set $A \Delta B$ is the set $\{ x \mid \text{exactly one of } x \in A \text{ and } x \in B \text{ is true} \}$.

Power Sets

The power set of a set S, denoted $\wp(S)$, is the set of all subsets of S. Using set-builder notation, this is the set $\wp(S) = \{ U \mid U \subseteq S \}$. Cantor's Theorem states that $|S| < |\wp(S)|$ for every set S.

Special Sets

The set $\emptyset = \{ \}$ is the empty set containing no elements.

The set $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots \}$ is the set of all natural numbers. We treat 0 as a natural number.

The set $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of all integers.

The set $\mathbb{R}$ consists of all the real numbers. The set $\mathbb{Q}$ consists of all rational numbers.

Cardinality

The cardinality of a finite set S (denoted $|S|$) is the natural number equal to the number of elements in that set. The cardinality of $\mathbb{N}$ (denoted $|\mathbb{N}|$) is $\aleph_0$ (pronounced "aleph-nought"). Two sets have the same cardinality iff there is a way of pairing up each element of the two sets such that every element of each set is paired with exactly one element of the other set.